THE NON-COMMUTATIVE SCHEME HAVING A FREE ALGEBRA AS A HOMOGENEOUS COORDINATE RING

S. PAUL SMITH

Abstract. Let k be a field and TV the tensor algebra on a finite-dimensional k-vector space V. This paper proves that the quotient category $\mathsf{QGr}(TV) := \mathsf{Gr}(TV)/\mathsf{Fdim}$ of graded TV-modules modulo those that are unions of finite dimensional modules is equivalent to the category of modules over the direct limit of matrix algebras, $\varinjlim_r M_n(k)^{\otimes r}$. Non-commutative algebraic geometry associates to a graded algebra A a "non-commutative scheme" $\operatorname{Proj}_{nc} A$ that is defined implicitly by declaring that the category of "quasi-coherent sheaves" on $\operatorname{Proj}_{nc} A$ is $\mathsf{QGr}\, A$. When A is coherent and $\mathsf{gr} A$ its category of finitely presented graded modules, $\mathsf{qgr}\, A := \mathsf{gr} A/\mathsf{fdim}$ is viewed as the category of "coherent sheaves" on $\operatorname{Proj}_{nc} A$. We show that when $\dim V \geq 2$, $\mathsf{qgr}(TV)$ has no indecomposable objects, no noetherian objects, and no simple objects. Moreover, every short exact sequence in $\mathsf{qgr}(TV)$ splits.

We also prove $\mathsf{QGr}(TV) \equiv \mathsf{Gr} L$ where L is the Leavitt algebra on $2 \dim V$ generators that embeds as a dense subalgebra of the Cuntz algebra $\mathcal{O}_{\dim V}$.

1. Introduction

1.1. Let n be a positive integer.

Throughout this paper k is a field and

$$R := k\langle x_0, x_1, \ldots, x_n \rangle$$

is the free algebra on $n+1$ variables with $\mathbb{Z}$-grading given by declaring that $\deg x_i = 1$ for all i. This paper concerns the categories of coherent and quasi-coherent "sheaves" on the "non-commutative scheme"

$$\mathbb{X}^n := \operatorname{Proj}_{nc} k\langle x_0, x_1, \ldots, x_n \rangle$$

with "homogeneous coordinate ring" R.

The "scheme" $\operatorname{Proj}_{nc} R$ is an imaginary object: there is no underlying topological space endowed with a sheaf of rings. Rather one declares that the category of "quasi-coherent sheaves" on $\operatorname{Proj}_{nc} R$ is the quotient category

$$\mathsf{Qcoh}(\operatorname{Proj}_{nc} R) := \mathsf{QGr}\, R := \frac{\mathsf{Gr} R}{\mathsf{Fdim} R}$$

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where $\mathsf{Gr}R$ is the category of $\mathbb{Z}$-graded left R-modules with degree-preserving homomorphisms and $\mathsf{Fdim}R$ its full subcategory of direct limits of finite-dimensional modules. The imaginary space $\operatorname{Proj}_{nc} R$ manifests itself through the category $\mathsf{Qcoh}(\operatorname{Proj}_{nc} R)$.

The category $\mathsf{QGr}\, R$ and its full subcategory of finitely presented objects, $\mathsf{qgr}\, R$, is the focus of this paper.[1] We think of $\mathsf{qgr}\, R$ as the category of "coherent sheaves" on $\operatorname{Proj}_{nc} R$.

1.2. The notations $\mathsf{QGr}\, R$ and $\mathsf{Qcoh}\mathbb{X}^n$ are interchangeable as are the notations $\mathsf{qgr}\, R$ and $\mathsf{coh}\mathbb{X}^n$. The reader may adopt either so as to reinforce either an algebraic or geometric perspective.

1.3. We write
$$\pi^* : \mathsf{Gr}R \to \mathsf{QGr}\, R$$
for the quotient functor and define $\mathcal{O} := \pi^* R$. We call $\mathcal{O}$ a structure sheaf for $\operatorname{Proj}_{nc} R$.

For the commutative polynomial ring $k[x_0, \ldots, x_n]$, $\mathsf{QGr}\, k[x_0, \ldots, x_n]$ is equivalent to $\mathsf{Qcoh}\mathbb{P}^n$ and π^* "is" the functor usually denoted $M \mapsto \widetilde{M}$ in algebraic geometry texts and $\pi^*(k[x_0, \ldots, x_n])$ *is* the structure sheaf $\mathcal{O}_{\mathbb{P}^n}$.

1.4. The twist functor on $\mathsf{Gr}R$, denoted $M \rightsquigarrow M(n)$, is defined by $M(n)_i := M_{n+i}$ with the same action of R. The subcategory $\mathsf{Fdim}R$ is stable under twisting so there is an induced functor on $\mathsf{QGr}\, R$ that we denote by $\mathcal{F} \rightsquigarrow \mathcal{F}(n)$ and call the Serre twist.

1.5. **The main results.** We define the direct limit algebra
$$S := \varinjlim_i M_{n+1}(k)^{\otimes i}$$
where the maps in the directed system are $a_1 \otimes \cdots \otimes a_i \mapsto 1 \otimes a_1 \otimes \cdots \otimes a_i$. The ring S is coherent so finitely presented S-modules form an abelian category.

Theorem 1.1. *There is an equivalence of categories*
$$\operatorname{Hom}_{\mathsf{QGr}\, R}(\mathcal{O}, -) : \mathsf{QGr}\, R \equiv \mathsf{Mod}_r S,$$
the category of right S-modules. The equivalence sends $\mathcal{O}$ to S, i.e., $S = \operatorname{Hom}_{\mathsf{QGr}\, R}(\mathcal{O}, \mathcal{O})$. Furthermore, the equivalence restricts to an equivalence
$$\mathsf{qgr}\, R \equiv \mathsf{mod}_r S,$$
the category of finitely presented right S-modules.

The ring S is anti-isomorphic to itself so it doesn't really matter whether we choose to work with left or right S-modules.

The key to Theorem 1.1 is the following preliminary result.

[1]An object $\mathcal{M}$ in an additive category A is finitely presented if $\operatorname{Hom}_{\mathsf{A}}(\mathcal{M}, -)$ commutes with direct limits; is finitely generated if whenever $\mathcal{M} = \sum \mathcal{M}_i$ for some directed family of subobjects $\mathcal{M}_i$ there is an index j such that $\mathcal{M} = \mathcal{M}_j$; is coherent if it is finitely presented and all its finitely generated subobjects are finitely presented.

Theorem 1.2. *$\mathcal{O}$ is a finitely generated, projective, generator in $\mathsf{Qcoh}\mathbb{X}^n$.*

As the next result emphasizes, the categories $\mathsf{coh}\mathbb{X}^n$ and $\mathsf{Qcoh}\mathbb{X}^n$ are unlike the categories of coherent and quasi-coherent sheaves over quasi-projective schemes.

Theorem 1.3. *Suppose $n \geq 1$.*

1. *There are no indecomposable objects in $\mathsf{coh}\mathbb{X}^n$, hence no simple objects, and therefore no noetherian objects other than 0.*
2. *Every short exact sequence in $\mathsf{coh}\mathbb{X}^n$ splits.*
3. *Every object in $\mathsf{coh}\mathbb{X}^n$ is isomorphic to a finite direct sum of various $\mathcal{O}(i)$s with finite multiplicities.*
4. *If $\mathcal{F}, \mathcal{G} \in \mathsf{coh}\mathbb{X}^n$ are non-zero, then $\dim_k \operatorname{Hom}_{\mathbb{X}^n}(\mathcal{F}, \mathcal{G}) = \infty$.*
5. *The Grothendieck group of the abelian category $\mathsf{coh}\mathbb{X}^n$ is isomorphic to $\mathbb{Z}\big[\frac{1}{n+1}\big]$ as an additive group.*

The reason $\mathsf{coh}\mathbb{X}^n$ and $\mathsf{Qcoh}\mathbb{X}^n$ behave so differently from categories of (quasi-)coherent sheaves on quasi-projective schemes is that $\mathcal{O}(-1)$ is a non-trivial direct summand of $\mathcal{O}$. Since $R_{\geq 1}$ is isomorphic to $R(-1)^{\oplus(n+1)}$,

$$\mathcal{O} \cong \mathcal{O}(-1)^{\oplus(n+1)}.$$

This behavior also occurs for the graded algebras $k\langle x, y\rangle/(y^{r+1})$ in [15].

Section 2 of [15] establishes some general results that are applied in the present paper to the free algebra: almost everything in the present paper is a rather simple consequence of those more general results.

1.6. **Connection to Leavitt algebras and Cuntz algebras.** Let L be the Leavitt algebra generated by the entries in the row vectors $\underline{x} = (x_0, \ldots, x_n)$ and $\underline{x}^* = (x_0^*, \ldots, x_n^*)$ subject to the relations

$$\underline{x}^* \underline{x}^{\mathsf{T}} = 1 \qquad \text{and} \qquad \underline{x}^{\mathsf{T}} \underline{x}^* = I_{n+1},$$

the $(n+1) \times (n+1)$ identity matrix. Give L a $\mathbb{Z}$-grading by $\deg x_i = 1$ and $\deg x_i^* = -1$. The connection between $\mathsf{QGr}\, R$ and L is made in the following result which is proved in section 4.

Theorem 1.4. *The algebra L is strongly graded, $L_0 \cong S$, and there is an equivalence of categories*

$$\mathsf{QGr}\, R \equiv \mathsf{Gr} L \equiv \mathsf{Mod} L_0.$$

The proof makes use of the following facts: L is the universal localization of R that inverts the homomorphism $R(-1)^{n+1} \to R$ whose cokernel is $R/R_{\geq 1}$; L is flat as a right R-module; if $M \in \mathsf{Gr} R$ and $L \otimes_R M = 0$, then $M \in \mathsf{Fdim} R$. The first of these facts is well-known; the second is proved in [1]; a version of the third for modules in $\mathsf{gr} R$ is proved in [1].

In [5], Cuntz defined a class of C*-algebras $\mathcal{O}_{n+1}$, $n \geq 1$, generated by the "same" elements and relations for L. The algebra L is a dense subalgebra of $\mathcal{O}_{n+1}$.

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2. Coherent sheaves on $\operatorname{Proj}_{nc} TV$

Let V be a k-vector space of dimension $d = n + 1$ and

$$R := TV = k\langle x_0, \ldots, x_n\rangle$$

be the tensor algebra on V with $\mathbb{Z}$-grading $R_i := V^{\otimes i}$.

2.1. **Graded modules over the free algebra.** A ring is left coherent if all its finitely generated left ideals are finitely presented. Since TV is anti-isomorphic to itself we can dispense with the adjectives *left* and *right* when discussing properties of TV like coherence.

The important property for us, indeed an equivalent characterization of coherence, is that the category of finitely presented left modules over a left coherent ring is abelian.

The following facts are well-known.

Proposition 2.1. *Let V be a vector space of finite dimension $d \geq 1$.*

1. *Every left ideal in TV is free.*
2. *TV is coherent and has global dimension one.*
3. *Every finitely generated projective R-module is free.*
4. *If $d \geq 2$, TV has exponential growth: $H(R,t) = (1 - dt)^{-1}$.*
5. *If $d \geq 2$, TV is not noetherian.*
6. *$R_{\geq i}$ is isomorphic to $R(-i)^{d^i}$, the free R-module of rank d^i with basis in degree i.*

Suppose N is a graded R-module. There is an exact sequence $L \to M \to N \to 0$ in $\mathsf{Mod}R$ with L and M finitely generated if and only if there is an exact sequence $L' \to M' \to N \to 0$ in $\mathsf{Gr}R$ with L' and M' generated by a finite number of homogeneous elements.

We define

$$\mathsf{gr}R := \text{the category of finitely presented graded } R\text{-modules.}$$

This is an abelian category because R is coherent.

Proposition 2.2. *Let $R = TV$ and $M \in \mathsf{QGr}\, R$.*

(1) *M is graded-coherent if and only if for all $i \gg 0$,*
$$M_{\geq i} \cong R(-i)^{t_i}$$
for some integer t_i depending on i.
(2) *If $0 \to L \to M \to N \to 0$ is an exact sequence in $\mathsf{gr}R$, then $0 \to L_{\geq i} \to M_{\geq i} \to N_{\geq i} \to 0$ splits for $i \gg 0$.*
(3) *If $M \in \mathsf{gr}R$, then M has a largest finite dimensional graded submodule.*

Proof. (1) Suppose M is finitely presented. Then there is an exact sequence $0 \to F' \to F \to M \to 0$ in $\mathsf{gr}R$ with F and F' finitely generated graded free R-modules. Since F', F, and M, are finitely generated graded modules, $F'_{\geq i}$, $F_{\geq i}$, and $M_{\geq i}$, are generated as R-modules by F'_i, F_i, and M_i, respectively for all sufficiently large i. But $R_{\geq m} \cong R(-m)^{d^m}$ so
$$\big(0 \to F' \to F \to M \to 0\big)_{\geq i} \;=\; \big(0 \to R(-i)^r \to R(-i)^s \to M_{\geq i} \to 0\big)$$
for $i \gg 0$. Every degree-zero homomorphism $R(-i)^r \to R(-i)^s$ splits so $M_{\geq i} \cong R(-i)^{s-r}$.

The converse is trivial.

(2) By (1), $N_{\geq i}$ is free for $i \gg 0$, hence the splitting.

(3) Since R is a domain its only finite dimensional submodule is zero. It now follows from (1) that the only finite dimensional submodule of $M_{\geq i}$ is the zero submodule for $i \gg 0$. There is therefore an integer n such that every finite dimensional submodule of M is contained in $\sum_{j=-i}^{i} M_j$. But M is finitely generated so $\sum_{j=-i}^{i} M_j$ has finite dimension. Hence the sum of all finite dimensional submodules of M has finite dimension, and that sum is therefore the largest finite dimensional graded submodule of M. □

2.2. We write $\mathsf{fdim}R$ for $\mathsf{Fdim}R \cap \mathsf{gr}R$. Thus $\mathsf{fdim}R$ is the full subcategory of $\mathsf{Gr}R$ consisting of the finite dimensional submodules. We define the category of "coherent sheaves" on $\mathbb{X}^n$ by
$$\mathsf{coh}\mathbb{X}^n = \mathsf{qgr}\, R := \frac{\mathsf{gr}R}{\mathsf{fdim}R}.$$
Since Fdim satisfies condition (2) of [8, Prop. A.4, p. 113] with respect to the Serre subcategory fdim of $\mathsf{gr}R$, Fdim is localizing of finite type which then allows us to apply [8, Prop. A.5, p. 113] and so conclude that $\mathsf{coh}\mathbb{X}^n$ consists of finitely presented objects in $\mathsf{Qcoh}\mathbb{X}^n$ and every object in $\mathsf{Qcoh}\mathbb{X}^n$ is a direct limit of objects in $\mathsf{coh}\mathbb{X}^n$.

Proposition 2.3. *Every short exact sequence in $\mathsf{qgr}\, R$ splits.*

Proof. By [6, Cor. 1, p. 368], every short exact sequence in $\mathsf{qgr}\, R$ is of the form
$$0 \longrightarrow \pi^*L \xrightarrow{\pi^*f} \pi^*M \xrightarrow{\pi^*g} \pi^*N \longrightarrow 0$$
where $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ is an exact sequence in $\mathsf{gr}R$. But
$$\pi^*\big(0 \to L \to M \to N \to 0\big) \;\cong\; \pi^*\big(0 \to L_{\geq i} \to M_{\geq i} \to N_{\geq i} \to 0\big)$$

for all n and $N_{\geq i}$ is free for $i \gg 0$ so the sequence $0 \to L_{\geq i} \to M_{\geq i} \to N_{\geq i} \to 0$ splits for $i \gg 0$. Applying π^* to a split exact sequence yields a split exact sequence, whence the result. $\square$

Corollary 2.4. *Let $\mathcal{F} \in \mathsf{qgr}\, R$. If $i \gg 0$ there is an integer r, depending on i, such that*

$$\mathcal{F} \cong \mathcal{O}(i)^r.$$

Proof. There is an $M \in \mathsf{gr} R$ such that $\mathcal{F} = \pi^* M$. But $\dim_k M/M_{\geq i} < \infty$ so $\pi^* M \cong \pi^* M_{\geq i}$ for all $i \gg 0$. Now apply Proposition 2.2(1). $\square$

Corollary 2.5. *Every object in $\mathsf{qgr}\, R$ is injective and projective. Furthermore, $\mathcal{O}$ is projective as an object in $\mathsf{QGr}\, R$.*

Proof. It follows from Proposition 2.3 that every object in $\mathsf{qgr}\, TV$ is injective and projective in $\mathsf{qgr}\, TV$. Since every left ideal of R is free it is a tautology that every graded left ideal that has finite codimension in R contains a *free* graded left ideal that has finite codimension in R. Hence [15, Prop. 2.7] applies and tells us that $\mathcal{O}$ is projective in $\mathsf{QGr}\, R$. $\square$

2.3. Since $R_{\geq i} \cong R(-i)^{\oplus (\dim V)^i}$ has finite codimension in R, [15, Prop. 2.7 and Thm. 2.8] may be applied to yield the next two results.

See Footnote 1 for the definition of finitely generated, finitely presented, and coherent, objects in an abelian category.

Proposition 2.6. *Let A be a left graded-coherent ring and write $\mathcal{O}$ for the image of A in $\mathsf{QGr}\, A$. Then*

1. *$\mathsf{QGr}\, A$ is a locally coherent category;*
2. *the full subcategory of $\mathsf{QGr}\, A$ consisting of the finitely presented objects is equivalent to $\mathsf{qgr}\, A$;*
3. *$\mathcal{O}$ is coherent: i.e., $\operatorname{Hom}_{\mathsf{QGr}\, A}(\mathcal{O}, -)$ commutes with direct limits;*
4. *$\mathcal{O}$ is finitely generated.*

Theorem 2.7.

1. *$\mathcal{O}$ is a progenerator in $\mathsf{QGr}\, R$.*
2. *The functor $\operatorname{Hom}(\mathcal{O}, -)$ is an equivalence from the category $\mathsf{QGr}\, R$ to the category of right modules over the endomorphism ring $\operatorname{End}_{\mathsf{QGr}\, R} \mathcal{O}$.*
3. *$\operatorname{End}_{\mathsf{QGr}\, R} \mathcal{O} \cong \varinjlim \operatorname{End}_{\mathsf{Gr} R}(R_{\geq i})$, the direct limit of the directed system*

$$\cdots \longrightarrow \operatorname{End}_{\mathsf{Gr} R}(R_{\geq i}) \xrightarrow{\theta_i} \operatorname{End}_{\mathsf{Gr} R}(R_{\geq i+1}) \longrightarrow \cdots \tag{2-1}$$

of k-algebras in which $\theta_i(f) = f|_{R_{\geq i+1}}$.

We will determine this direct limit in section 3.

2.4.

Lemma 2.8. *If* $\dim V = d \geq 1$, *then*

$$\mathcal{O} \cong \mathcal{O}(-1)^{\oplus d} \cong \mathcal{O}(-2)^{\oplus d^2} \cong \cdots$$

Proof. From the exact sequence $0 \to R_{\geq 1} \to R \to k \to 0$ we see that $\pi^* R \cong \pi^* R_{\geq 1}$. But $R_{\geq 1} \cong R \otimes_k V \cong R(-1)^d$ so $\pi^* R_{\geq 1} \cong \mathcal{O}(-1)^d$. Hence $\mathcal{O} = \pi^* R \cong \mathcal{O}(-1)^d$. The result now follows by induction. $\square$

Lemma 2.8 implies that for every $\mathcal{F} \in \mathsf{qgr}\, R$ and every $r \geq 1$,

$$\mathcal{F} \cong \mathcal{F}(-r)^{\oplus d^r}.$$

Corollary 2.9. *Let* $d = \dim V \geq 1$. *If* $\mathcal{F} \in \mathsf{qgr}\, R$ *is non-zero, then for all integers* $r \geq 0$ *there is an injective ring homomorphism*

$$M_{d^r}(k) \to \operatorname{End} \mathcal{F}.$$

Corollary 2.10. *Suppose* $\dim V \geq 2$. *Then*

1. *the only noetherian object in* $\mathsf{qgr}\, R$ *is the zero object;*
2. *there are no simple objects in* $\mathsf{qgr}\, R$*;*
3. *there are no indecomposable objects in* $\mathsf{qgr}\, R$*;*
4. $\dim_k \operatorname{Hom}_{\mathsf{QGr}\, R}(\mathcal{F}, \mathcal{G}) = \infty$ *for all non-zero objects* $\mathcal{F}$ *and* $\mathcal{G}$ *in* $\mathsf{qgr}\, R$.

Proof. (2) By Schur's lemma the endomorphism ring of a simple object is a division algebra but Corollary 2.9 says that endomorphism rings of objects in $\mathsf{coh}\mathbb{X}^n$ are never division rings when $\dim V \geq 2$. Hence $\mathsf{coh}\mathbb{X}^n$ has no simple objects. Part (1) follows immediately because a non-zero noetherian object has at least one maximal subobject and hence a simple quotient.

(3) See the remark after Lemma 2.8.

(4) Since $\operatorname{Hom}_{\mathbb{X}^n}(\mathcal{F}, \mathcal{G})$ is an $\operatorname{End} \mathcal{G}$-$\operatorname{End} \mathcal{F}$-bimodule it is a module over the matrix algebra $M_{d^r}(k)$ for all $r \neq 0$. It therefore suffices to show that $\operatorname{Hom}_{\mathbb{X}^n}(\mathcal{F}, \mathcal{G})$ is non-zero. By Lemma 2.8 there is a monic map $\mathcal{O}(-1) \to \mathcal{O}$ and an epic map $\mathcal{O} \to \mathcal{O}(-1)$. Because the twist (1) is an auto-equivalence it follows (using compositions of twists of the monic and epic maps just mentioned) that there is a monic map $\mathcal{O}(i) \to \mathcal{O}(j)$ whenever $i \leq j$ and an epic map $\mathcal{O}(j) \to \mathcal{O}(i)$ whenever $j \geq i$. It now follows from Corollary 2.4 that $\operatorname{Hom}_{\mathbb{X}^n}(\mathcal{F}, \mathcal{G}) \neq 0$. $\square$

3. A direct limit of matrix algebras

We will now show that the endomorphism ring of $\mathcal{O}$ is isomorphic to the ring S we are about to define.

Let $S_i := M_{n+1}(k)^{\otimes i}$ and define

$$\theta_i : S_i \to S_{i+1} = M_{n+1}(k) \otimes S_i \qquad \text{by} \qquad \theta_i(a) = 1 \otimes a. \tag{3-1}$$

The homomorphisms θ_i determine a directed system and we define

$$S := \varinjlim_i S_i.$$

We write $\mathsf{Mod}S$ for the category of right S-modules and $\mathsf{mod}S$ for its full category of finitely presented S-modules.

Theorem 3.1. *If S is the ring above, then $\operatorname{End}_{\mathbb{X}^n}\mathcal{O} \cong S$, the functor $\operatorname{Hom}_{\mathbb{X}^n}(\mathcal{O},-)$ is an equivalence of categories*

$$\mathsf{Qcoh}\mathbb{X}^n \equiv \mathsf{Mod}S$$

sending $\mathcal{O}$ to S_S, and $\operatorname{Hom}_{\mathbb{X}^n}(\mathcal{O},-)$ restricts to an equivalence

$$\mathsf{coh}\mathbb{X}^n \equiv \mathsf{mod}S.$$

Proof. Write $R = TV$ as in section 2. By the definition of morphisms in a quotient category,

$$\operatorname{End}_{\mathbb{X}^n}\mathcal{O} = \operatorname{Hom}_{\mathsf{QGr}\,R}(\pi^*R,\pi^*R) = \varinjlim \operatorname{Hom}_{\mathsf{Gr}R}(R', R/R'') \tag{3-2}$$

where R' runs over all graded left ideals in R such that $\dim_k(R/R') < \infty$ and R'' runs over all graded left ideals in R such that $\dim_k R'' < \infty$.

By Theorem 2.7 (see [15, Sect. 2] for a fuller explanation), this reduces to

$$\operatorname{End}_{\mathsf{QGr}\,R}\mathcal{O} = \varinjlim_i \operatorname{Hom}_{\mathsf{Gr}R}(R_{\geq i}, R_{\geq i}).$$

As a left R-module, $R_{\geq i} \cong R \otimes_k V^{\otimes i}$ where V is placed in degree 1. There is a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_k(V^{\otimes i}, V^{\otimes i}) \arrow[r, "\psi_i"] \arrow[d, "\rho_i"'] & \operatorname{Hom}_k(V^{\otimes i+1}, V^{\otimes i+1}) \arrow[d, "\rho_{i+1}"] \\
\operatorname{Hom}_{\mathsf{Gr}R}(R_{\geq i}, R_{\geq i}) \arrow[r, "\theta_i"'] & \operatorname{Hom}_{\mathsf{Gr}R}(R_{\geq i+1}, R_{\geq i+1})
\end{tikzcd}$$

in which

$$\rho_i(f)(r \otimes v) = rf(v) \qquad \text{for all } r \in R \text{ and } v \in V^{\otimes i}$$

and

$$\psi_i(f)(a_0 \otimes a_1 \otimes \cdots \otimes a_i) = a_0 \otimes f(a_1 \otimes \cdots \otimes a_i).$$

But ρ_i is an isomorphism so

$$\operatorname{End}_{\mathbb{X}^n}\mathcal{O} \cong \varinjlim_i \operatorname{End}_k V^{\otimes i} \cong \varinjlim_i M_{n+1}(k)^{\otimes i}.$$

The proof is complete. □

The following properties of S are either obvious or well-known (see [7]).

Proposition 3.2.

(1) *S is a simple ring;*
(2) *S has no finite dimensional modules other than 0.*
(3) *Every finitely generated left ideal of S is generated by an idempotent.*
(4) *S is a von Neumann regular ring.*
(5) *S is left and right coherent.*
(6) *Every left S-module is flat.*

(7) *Every finitely generated left S-module is projective.*
(8) $K_0(S) \cong \mathbb{Z}\big[\frac{1}{n+1}\big]$ *via an isomorphism sending* $[S]$ *to 1.*

Proof. (8) Since $K_0(-)$ commutes with direct limits and $K_0(S_i) \cong \mathbb{Z}$ with $[S_i] = n+1$ under the isomorphism, $K_0(S)$ is the direct limit of the directed system

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{n+1} \mathbb{Z} \xrightarrow{n+1} \cdots .$$

This direct limit is obviously isomorphic to $\mathbb{Z}[\frac{1}{n+1}]$. □

It seems worthwhile to determine the Grothendieck group of $\mathsf{qgr}\, R$ independently of the equivalence of categories, i.e., without appealing to part (8) of Proposition 3.2. The next result does this.

Proposition 3.3. *As an additive group, the Grothendieck group of* $\mathsf{coh}\mathbb{X}^n$ *is isomorphic to* $\mathbb{Z}[\frac{1}{n+1}]$ *with* $[\mathcal{O}(i)] \longleftrightarrow (n+1)^i$.

Proof. Since $\mathsf{fdim}R$ is a Serre subcategory of $\mathsf{gr}R$ there is an exact sequence

$$K_0(\mathsf{fdim}R) \to K_0(\mathsf{gr}R) \to K_0(\mathsf{coh}\mathbb{X}^n) \to 0.$$

It is clear that $K_0(\mathsf{gr}R) \cong \mathbb{Z}[t^{\pm 1}]$ via $R(-i) \leftrightarrow t^i$. From the exact sequence $0 \to R(-1)^{n+1} \to R \to k \to 0$ we obtain $[k] = [R] - (n+1)[R(-1)] = 1-(n+1)t$. Modules in $\mathsf{fdim}R$ are finite dimensional so have compositions series in which the composition factors are of the form $k(i)$ for various integers i. Hence, by dévissage, $K_0(\mathsf{fdim}R) \cong K_0(\mathsf{gr}k)$ which is isomorphic as an additive group to $\mathbb{Z}[t^{\pm 1}]$ with $k(i) \longleftrightarrow t^{-i}$. The image of the map $K_0(\mathsf{fdim}R) \to K_0(\mathsf{gr}R)$ is therefore the ideal in $\mathbb{Z}[t^{\pm 1}]$ generated by the image of $[k]$. Therefore

$$K_0(\mathsf{coh}\mathbb{X}^n) \cong \frac{\mathbb{Z}[t]}{(1-(n+1)t)} \cong \mathbb{Z}\Big[\frac{1}{n+1}\Big].$$

This completes the proof. □

Corollary 3.4. *If* $m \neq n$, *then* $\mathbb{X}^m \ncong \mathbb{X}^n$.

It is reasonable to define $\operatorname{rank} \mathcal{O}(i) = (n+1)^i$.

4. The relation to the Leavitt algebra $L(1, n+1)$ and the Cuntz algebra $\mathcal{O}_{n+1}$

As before R is the free algebra $k\langle x_0, \ldots, x_n\rangle$.

The main result in this section is that $\mathsf{QGr}\, R \equiv \mathsf{Gr}L \equiv \mathsf{Mod}L_0$ where $L = L(1, n+1)$ is the finitely generated $\mathbb{Z}$-graded algebra defined below. It is well-known that L_0 is the algebra we have called S in the earlier part of this paper.

4.1. **The Leavitt algebra.** The Leavitt algebra $L = L(1, n+1)$, first defined in [9], is the k-algebra generated by elements $x_0, \ldots, x_n, x_0^*, \ldots, x_n^*$ subject to the relations

$$x_i x_i^* = 1 = x_0^* x_0 + \cdots + x_n^* x_n \qquad \text{and} \qquad x_i x_j^* = 0 \ \text{ if } i \neq j. \tag{4-1}$$

A more meaningful definition is that L is the universal localization [3, Sect. 7.2] of R inverting the injective homomorphism

$$\iota : R^{n+1} \to R, \qquad (r_0, \ldots, r_n) \mapsto r_0 x_0 + \cdots + r_n x_n.$$

Since ι is right multiplication by $(x_0, \ldots, x_n)^{\mathsf{T}}$, the formal inverse of ι is right multiplication by $(x_0^*, \ldots, x_n^*)$ where

$$(x_0, \ldots, x_n)^{\mathsf{T}}(x_0^*, \ldots, x_n^*) = I_{n+1} \quad \text{and} \quad (x_0^*, \ldots, x_n^*)(x_0, \ldots, x_n)^{\mathsf{T}} = 1$$

and I_{n+1} is the identity matrix.

4.2. **The Cuntz algebra.** The Cuntz algebra $\mathcal{O}_{n+1}$ [5] is the universal C^*-algebra generated by elements $x_0, \ldots, x_n$ subject to the relations (4-1). It is well-known that L embeds in $\mathcal{O}_{n+1}$ as a dense subalgebra. Much of the work in Cuntz's paper [5] involves purely algebraic calculations carried out inside L.

4.3. One anticipates a relation between $\mathsf{Gr}L$ and $\mathsf{QGr}\,R$ because the fact that $\mathrm{id}_L \otimes \iota$ is an isomorphism implies that

$$0 = L \otimes_R \operatorname{coker}(\iota) = L \otimes_R (R/R_{\geq 1}).$$

It follows that $L \otimes_R -$ kills all finite dimensional graded R-modules, and hence all modules in $\mathsf{Fdim}R$.

4.4. We make L a $\mathbb{Z}$-graded algebra by defining $\deg x_i = 1$ and $\deg x_i^* = -1$ for all i. The canonical map $R \to L$ is a homomorphism of graded rings and is injective. It is well-known, and not hard, to show that if $r > 0$, then $L_r = x_0^r L_0$ and $L_{-r} = L_0 (x_0^*)^r$ (see, e.g., [5, Sect. 1.6]). It follows from this that L is strongly graded, i.e., $L_j L_{-j} = L_0$ for all integers j, and therefore

$$\mathsf{Gr}L \equiv \mathsf{Mod}L_0$$

where the functor giving the equivalence sends a graded module M to its degree-zero component M_0.

Proposition 4.1 (Cuntz)**.** [5, Prop. 1.4] $L_0 \cong S$.

4.5. For $r \geq 1$, let X_r be the set of words of length r in the letters x_i, $0 \leq i \leq n$, and X_∞ the union of all X_r, $r \geq 0$. Cuntz [5, Lem. 1.3] shows that $L = \operatorname{span}\{w^* w' \mid w, w' \in X_\infty\}$.

4.6. The next result was proved in [1, Prop. 2.1] but its utility is such that it seems useful to give a more direct proof.

Proposition 4.2. *Let $R = k\langle x_0, \dots, x_n\rangle$ and define L as above. The ring L is flat as a right R-module.*

Proof. We will show L is an ascending union of finitely generated free right R-modules. Let

$$F_r = \sum_{w \in X_r} w^* R. \tag{4-2}$$

Suppose

$$\sum_{w \in X_r} w^* r_w = 0$$

for some elements $r_w \in R$. Let $z \in X_r$. Then $zw^* = \delta_{w,z}$ so $r_z = 0$. It follows that all the r_ws are zero and the sum in (4-2) is therefore a direct sum. Because $ww^* = 1$, each w^*R is a free R-module. Hence F_r is free.

Since $w^* = \sum_{i \in I} w^* x_i^* x_i$, $F_r \subset F_{r+1}$. Since L is spanned by elements w^*w', L is the ascending union of the F_rs and therefore flat. □

A version of the following result for finitely presented not-necessarily-graded modules is given in [1, Thm. 5.1]. Our proof differs in spirit from that in [1].

Proposition 4.3. *Let R and L be as above and $M \in \mathsf{Gr}R$. Then $L \otimes_R M = 0$ if and only if $M \in \mathsf{Fdim}R$.*

Proof. We observed in section 4.3 that $L \otimes_R M = 0$ if $M \in \mathsf{Fdim}R$.

To prove the converse suppose $L \otimes_R M = 0$. First we will show M is finite dimensional under the additional hypothesis that it is finitely presented. By Proposition 2.2, $M_{\geq i}$ is a free R-module for $i \gg 0$. If $M_{\geq i}$ is a *non-zero* free module, then $L \otimes_R M$ would contain a non-zero free L-module. This does not happen because $L \otimes_R M = 0$ so we deduce that $M_{\geq i} = 0$ for $i \gg 0$. Since M is finitely generated it is therefore finite dimensional.

Now let M be an arbitrary graded R-module such that $L \otimes_R M = 0$. To prove the proposition it suffices to show that $\dim_k Rm < \infty$ for all homogeneous $m \in M$. Let $m \in M$ be a homogeneous element. Since R is coherent M is a direct limit of finitely presented graded modules, say $M = \varinjlim M_\lambda$ where each M_λ is finitely presented. Let $\theta_\lambda : M_\lambda \to M$ be the canonical map and let $m_\lambda \in M_\lambda$ be a homogeneous element such that $\theta_\lambda(m_\lambda) = m$. Using θ_λ, there is a map $L \otimes_R Rm_\lambda \to L \otimes_R Rm$; but $L \otimes_R Rm = 0$ so the image of $1 \otimes m_\lambda$ in $L \otimes_R M_\nu$ is zero for some $\nu \gg i$. Let m_ν be the image of m_λ in M_ν. Then $L \otimes_R Rm_\nu = 0$. Since R is coherent and Rm_ν is a finitely generated submodule of M_ν, Rm_ν is finitely presented. The previous paragraph allows us to conclude that $\dim_k Rm_\nu < \infty$. But Rm is the image of Rm_ν in M so $\dim_k Rm < \infty$. □

A version of the next result for finitely presented not-necessarily-graded modules is given in [1, Thm. 5.1]. As with the previous result, the ideas in our proof differ from those in [1].

Theorem 4.4. *Let* $\pi^* : \mathsf{Gr}R \longrightarrow \mathsf{QGr}\,R$ *be the quotient functor and let* $i^* = L \otimes_R - : \mathsf{Gr}R \longrightarrow \mathsf{Gr}L$*. Then*

$$\mathsf{QGr}\,R \equiv \mathsf{Gr}L$$

via a functor $\alpha^* : \mathsf{QGr}\,R \to \mathsf{Gr}L$ *such that* $\alpha^*\pi^* = i^*$*.*

Proof. We already know i^* is exact and vanishes on $\mathsf{Fdim}R$ so, by the universal property of $\mathsf{QGr}\,R$, there is a unique functor $\alpha^* : \mathsf{QGr}\,R \to \mathsf{Gr}L$ such that $\alpha^*\pi^* = i^*$; furthermore, α^* is exact.

The forgetful functor $i_* : \mathsf{Gr}L \to \mathsf{Gr}R$ is exact and right adjoint to i^*. We will show that $\pi^* i_*$ is quasi-inverse to α^*. A diagram will help us keep track of the data:

$$\begin{tikzcd} \mathsf{Gr}R \arrow[r, "\pi^*"] \arrow[d, "i^*"'] & \mathsf{QGr}\,R \arrow[dl, "\alpha^*"] \\ \mathsf{Gr}L \arrow[u, bend left=60, "i_*"] & \end{tikzcd}$$

Since $R \to L$ is a universal localization it is an epimorphism in the category of rings. The multiplication map $L \otimes_R L \to L$ is therefore an isomorphism of L-bimodules. Thus, if $N \in \mathsf{Gr}L$, then

$$i^*i_*N = L \otimes_R N = L \otimes_R (L \otimes_L N) \cong N.$$

Therefore $\alpha^*(\pi^* i_*) = i^* i_* \cong \mathrm{id}_{\mathsf{Gr}R}$.

Let $M \in \mathsf{Gr}R$ and consider the exact sequence

$$0 \to \mathrm{Tor}_1^R(L/R, M) \to M \xrightarrow{f} L \otimes_R M = i_*i^*M \to (L/R) \otimes_R M \to 0 \tag{4-3}$$

where $f(m) = 1 \otimes m$. Since $L \otimes_R L \cong L$, $i^*(f)$ is an isomorphism. But i^* is exact so it vanishes on $\mathrm{Tor}_1^R(L/R, M)$ and $(L/R) \otimes_R M$. Thus, by Proposition 4.3, both these modules are in $\mathsf{Fdim}R$. Therefore π^* vanishes on them. Hence $\pi^*(f)$ is an isomorphism. In other words, the natural transformation $\pi^* \to \pi^* i_* i^*$ is an isomorphism.

In particular, $\pi^* \cong \pi^* i_* \alpha^* \pi^*$. By the universal property of $\mathsf{QGr}\,R$, there is a unique functor β^* such that the diagram

$$\begin{tikzcd} \mathsf{Gr}R \arrow[r, "\pi^*"] \arrow[d, "\pi^*"'] & \mathsf{QGr}\,R \arrow[dl, "\beta^*"] \\ \mathsf{QGr}\,R & \end{tikzcd}$$

commutes, i.e., $\pi^* = \beta^*\pi^*$. That β^* is, of course, $\mathrm{id}_{\mathsf{QGr}\,R}$. But $\pi^* \cong (\pi^* i_* \alpha^*)\pi^*$ so we conclude that $\pi^* i_* \alpha^* \cong \mathrm{id}_{\mathsf{QGr}\,R}$. This completes the proof that α^* and $\pi^* i_*$ are mutually quasi-inverse. $\square$

5. Remarks

5.1. Most non-commutative projective algebraic geometry to date involves non-commutative rings that are noetherian. See, for example, Artin and Zhang's paper [2] and the survey article of Stafford-Van den Bergh [17]. Two notable exceptions are (1) the rings $A := k\langle x_1, \ldots, x_n\rangle/(f)$ where f is a homogeneous quadratic element of rank $n \geq 3$ ([10], [11]) and (2) the non-commutative homogeneous coordinate rings appearing in Polishchuk's work ([12] and [13]) on non-commutative elliptic curves or, equivalently, non-commutative 2-tori endowed with a complex structure. The significance of the first is that $\mathsf{D}^b(\mathsf{QGr}\, A)$ is equivalent to the bounded derived category of representations of the generalized Kronecker quiver (i.e., that with two vertices and n parallel arrows from one to the other); $\operatorname{Proj}_{nc} A$ is viewed as a non-commutative analogue of the projective line. Polishchuk's work provides a beautiful and deep connection between non-commutative geometry based on operator algebras and non-commutative projective algebraic geometry.

The direct limit algebra S in Theorem 1.1 provides a further link. When the base field is $\mathbb{C}$ the norm-completion of S belongs to an important class of C^*-algebras, the AF-algebras (AF=approximately finite). Under the philosophy that non-commutative C^*-algebras correspond to "non-commutative topological spaces" AF-algebras are often viewed as corresponding to 0-dimensional spaces; see, for example, the paragraph at the foot of page 10 of Connes's book [4], although they also exhibit features of higher dimensional spaces. A prominent example is the AF-algebra associated to the space of Penrose tilings (see [4] and [15] for details).

5.2. **A homological remark.** A connected graded k-algebra A is said to be $\mathsf{Artin\text{-}Schelter\ regular}$ of dimension d if $\operatorname{Ext}^d_A(k, A) = k$ and $\operatorname{Ext}^i_A(k, A) = 0$ when $i \neq d$. Many of the proofs in non-commutative projective algebraic geometry work only for Artin-Schelter regular rings of finite Gelfand-Kirillov dimension.

The next result shows that TV is far from being Artin-Schelter regular when $\dim V \geq 2$.

Lemma 5.1. *Let R be the free algebra on d variables. Then there is an exact sequence*

$$0 \to R^{d^2-1} \to \underline{\operatorname{Ext}}^1_R(k, R) \to k(1)^d \to 0$$

of graded right R-modules.

Proof. Applying $\underline{\operatorname{Hom}}_R(-, R)$ to a minimal resolution $0 \to R(-1)^d \to R \to k \to 0$ of left R-modules produces the top row in the commutative diagram

of exact sequences

(5-1)
$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathrm{Hom}}(R,R) \arrow[r] \arrow[d, "\cong"] & \underline{\mathrm{Hom}}(R(-1)^d,R) \arrow[r] \arrow[d, "\cong"] & \underline{\mathrm{Ext}}^1_R(k,R) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & R \arrow[r, "f"'] & R(1)^d \arrow[r] & \underline{\mathrm{Ext}}^1_R(k,R) \arrow[r] & 0
\end{tikzcd}$$

in which $f(1) = (x_1, \dots, x_d)$ and $x_1, \dots, x_d$ is a basis for V.

Let $e_1 := (1, 0, \dots, 0), e_2 := (0, 1, 0, \dots, 0), \dots, e_d := (0, \dots, 0, 1)$ be the standard basis for R^d. Define the left R-module homomorphism $h : R \to (R^d)^{\oplus d}$ by $h(1) = (e_1, \dots, e_d)$. Let $g : R^d \to R(1)^d$ be the unique left R-module homomorphism such that $g(e_i) = x_i$ for all i. There is an exact sequence $0 \to R^d \xrightarrow{g} R(1) \to k(1) \to 0$. Let $(g, \dots, g) : (R^d)^{\oplus d} \to (R(1)^d)^{\oplus d}$ be the left R-module homomorphism defined by $(g, \dots, g)(u_1, \dots, u_d) = \big(g(u_1), \dots, g(u_d)\big)$ where $u_i \in R^d$. Then there is a commutative diagram

(5-2)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & R \arrow[r, equal] \arrow[d, "h"'] & R \arrow[d, "f"] & & \\
0 \arrow[r] & (R^d)^{\oplus d} \arrow[r, "{(g,\dots,g)}"] \arrow[d] & R(1)^{\oplus d} \arrow[r] \arrow[d] & k(1)^{\oplus d} \arrow[r] \arrow[d, equal] & 0 \\
 & R^{d^2-1} \arrow[d] & \underline{\mathrm{Ext}}^1_R(k,R) \arrow[d] & k(1)^d & \\
 & 0 & 0 & &
\end{tikzcd}$$

in which the columns are exact. The result now follows by applying the Snake Lemma to this diagram. □

The bottom row of (5-1) yields an exact sequence $0 \to \mathcal{O} \to \mathcal{O}(1)^d \to \mathcal{E} \to 0$ in $\mathsf{coh}\mathbb{X}^{d-1}$ (to be precise, since we started with left R-modules we should be replace the bottom row of (5-1) with the analogous exact sequence of left R-modules). By the lemma, $\mathcal{E} \cong \mathcal{O}^{d^2-1}$).

5.3. $\mathbb{X}^n$ **has only the trivial closed subspaces.** There is a notion of closed subspace in non-commutative algebraic geometry [18, Sect. 3.3]. Rosenberg [14, Prop. 6.4.1, p.127] proved that closed subspaces of an affine nc-space are in natural bijection with the two-sided ideals in a coordinate ring for it. The only two-sided ideals in S are the zero ideal and S itself so the only closed subspaces of $\mathbb{X}^n$ are the empty set and $\mathbb{X}^n$ itself.

This is a surprise because the free algebra contains a wealth of two-sided ideals. For example, the polynomial ring on $n+1$ variables is a quotient

of the free algebra $k\langle x_0, \dots, x_n\rangle$ so $\mathsf{Qcoh}\mathbb{P}^n$ is a full subcategory of $\mathsf{Qcoh}\mathbb{X}^n$ but coherent $\mathcal{O}_{\mathbb{P}^n}$-modules are not finitely presented as objects in $\mathsf{Qcoh}\mathbb{X}^n$.

5.4. In [16] we extend the ideas and results in this paper to path algebras of quivers: the free algebra is replaced by a path algebra kQ and the category of "quasi-coherent sheaves" on $\operatorname{Proj}_{nc}(kQ)$ is equivalent to the category of modules over a direct limit of semisimple k-algebras, namely $\varinjlim \operatorname{End}_{kI}(kQ_1)^{\otimes n}$ where I is the set of vertices and kQ_1 the linear span of the arrows.

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Department of Mathematics, Box 354350, Univ. Washington, Seattle, WA 98195

E-mail address: smith@math.washington.edu